Back paper

Number Theory

Instructor: Ramdin Mawia | Marks: **45** | Course: M1 | **Time:** June 5, 2026; 10:00–13:00.

Attempt any FIVE problems. Each question carries 10 marks. The maximum you can score is 45

1. Solve the congruence $7x^2 + x + 5 \equiv 0 \pmod{389}$, or prove that it has no solutions in integers. **10**
2. Find the last 2026 digits of $13^{10^{2025}}$. **10**
3. Describe all primes p for which 13 is a quadratic nonresidue mod p. **10**
4. Find an asymptotic formula for the sum **10**

   $$\sum_{n \leqslant x} \frac{\tau(n)}{n}$$

   with a bounded error term as $x \to \infty$. Here $\tau(n)$ denotes the divisor-counting function.
5. Let $d \neq 1$ be a squarefree integer, and let $K = \mathbb{Q}[\sqrt{d}]$. Prove that the ring of integers $\mathcal{O}_K$ in K is either $\mathbb{Z}[\sqrt{d}]$ or $\mathbb{Z}[(1+\sqrt{d})/2]$ depending on whether $d \not\equiv 1 \pmod 4$ or $d \equiv 1 \pmod 4$.) **10**
6. Prove that if $d > 1$ is squarefree but not a prime, then $\mathcal{O}_K$ is not a UFD, where $K = \mathbb{Q}[\sqrt{-d}]$. **10**